XVII. *On the Method of determining, from the real Probabilities of Life, the Values of contingent Reverſions in which Three Lives are involved in the Survivorſhip. By Mr.* William Morgan, *F. R. S.*

Read May 26, 1791.

HAVING been encouraged to the further purſuit of the doctrine of ſurvivorſhips by the very honourable manner in which my two former Papers on this ſubject were received by the Royal Society, I think it my duty to ſubmit the reſult of my labours to their conſideration. The ſolutions of ſome of the following problems might have been derived from thoſe which I have already communicated; but the direct inveſtigation of each ſeparate problem being certainly more ſatisfactory, and the rules obtained by this means in general more ſimple, I have conſidered no problem as connected with another, except the relation between them either immediately ariſes from the ſolution, or is neceſſary to prove the truth of it. Being anxious to render myſelf as conciſe as poſſible, I have been minute only in the inveſtigation of the firſt problem, and have done little more than ſtate the contingencies which will determine the ſurvivorſhip in the others. By the aſſiſtance, however, of theſe, and the operations which are detailed in my former Papers, the theorems which I have given may be deduced without much difficulty.

In order to prevent unneceſſary repetitions, it may not be improper to begin with explaining the different ſymbols which are uſed in the following pages.

A, B, C, denote the value of an annuity on the reſpective lives of A, B, or C.

D, denotes the value of S on the contingency of C's ſurviving A (by my 2d prob. Vol. LXXVIII).

E, denotes the ſame value on the contingency of B's ſurviving A, found by the ſame problem.

F, denotes the value of an annuity on a life one year younger than B.

G, denotes the value of the abſolute reverſion of S after the death of A.

H, denotes the value of an annuity on a life one year younger than A.

K, denotes the value of an annuity on a life one year younger than C.

L, denotes the value of an annuity on the longeſt of the three lives of A, B, and C.

M, denotes the value of S by the firſt problem on the contingency that A's life ſhall be the *firſt* that fails.

N, denotes the value of an annuity on a life one year older than A.

P, denotes the value of an annuity on a life one year older than B.

R, denotes the value of S on the contingency of B's dying after A (by my 3d prob. Vol. LXXVIII).

S, denotes the given ſum.

T, denotes the value of an annuity on a life one year older than C.

V, denotes the perpetuity.

W, denotes the value of S on the contingency of C's dying after A (by my 3d prob. Vol. LXXVIII).

α and a denote the number of perſons living in a table of obſervations at the ages of H and A.

s, t, u, w, &c. denote the number of perſons living at the end of the 1ſt, 2d, 3d, &c. years from the age of A.

β and b, denote the number of perſons living at the ages of F and B.

m, n, o, p, &c. denote the number of perſons living at the end of the 1ſt, 2d, 3d, &c. years from the age of B.

$\varkappa$ and c denote the number of perſons living at the ages of K and C.

d, e, f, g, &c. denote the number of perſons living at the end of the 1ſt, 2d, 3d, &c. years from the age of C.

$$\left.\begin{array}{l} a', a'', a''', \&c. \\ b', b'', b''', \&c. \\ c', c'', c''', \&c. \end{array}\right\}$$ denote the decrements of life at the end of the 1ſt, 2d, 3d, &c. years from the reſpective ages of A, B, and C.

r, denotes the value of £. 1 increaſed by its intereſt for a year.

The combinations of two or three of the ſeveral letters A, B, C, F, H, &c. denote the values of annuities on the joint continuance of two or three of thoſe reſpective lives.

PROBLEM I.

To determine the value of a given ſum, payable if A ſhould be the *firſt* that fails of the three lives A, B, and C.

SOLUTION.

In order to receive the given ſum in the firſt year, it is neceſſary that one or other of four events ſhould happen. 1ſt, That all the three lives ſhould fail, and that A ſhould die firſt. 2dly, That B ſhould die *after* A, and C live. 3dly, That C ſhould die *after* A, and B live. 4thly, That A only ſhould die, and B and C both live. Theſe ſeveral contingencies being expreſſed by $\frac{a' . \overline{b-m} . \overline{c-d}}{3abc}$, $\frac{a' . \overline{b-m} . d}{2abc}$, $\frac{a' . \overline{c-d} . m}{2abc}$, and $\frac{a' . dm}{abc}$, reſpectively, their ſum will be $= \frac{1}{abc} \times \overline{\frac{a'bc}{3} + \frac{a'md}{3} + \frac{a'mc}{6} + \frac{a'bd}{6}}$. In the ſecond and following years one or other of the ſame events muſt take place in order to receive the given ſum; that is, they muſt either all three die, A dying firſt; or only A and B muſt die, A dying firſt; or only A and C muſt die, A dying firſt; or only A muſt die, and B and C both live. The different fractions expreſſing thoſe four contingencies for the ſecond year being $\frac{a'' . \overline{m-n} . \overline{d-e}}{3abc}$, $\frac{a'' . \overline{m-n} . e}{2abc}$, $\frac{a'' . \overline{d-e} . n}{2abc}$, and $\frac{a'' . ne}{abc}$; for the third year $\frac{a''' . \overline{n-o} . \overline{e-f}}{3abc}$, $\frac{a''' . \overline{n-o} . f}{2abc}$, $\frac{a''' . \overline{e-f} . o}{2abc}$, and $\frac{a''' . of}{abc}$, and ſo on, the whole value of the given ſum will be $= \frac{S . a'}{abcr} \times \overline{\frac{bc}{3} + \frac{md}{3} + \frac{mc}{6} + \frac{bd}{6}} + \frac{S . a''}{abcr^2} \times \overline{\frac{md}{3} + \frac{ne}{3} + \frac{nd}{6} + \frac{me}{6}} + \frac{S . a'''}{abcr^3} \times \overline{\frac{ne}{3} + \frac{of}{3} + \frac{oe}{6} + \frac{nf}{6}} +$ &c. which are $= \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a''' . en}{r^3} + \text{\&c.}} + \frac{S}{6abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a''' . oe}{r^3} + \text{\&c.}} + \frac{S}{6abc} \times \overline{\frac{a'db}{r} + \frac{a''em}{r^2} + \frac{a''' . fn}{r^3} + \text{\&c.}} + \frac{S}{3abc} \times \overline{\frac{a\,md}{r} + \frac{a'' . en}{r^2} + \frac{a'''fo}{r^3} + \text{\&c.}}$ From the demonſtration of the problem in my laſt Paper * it appears, that the firſt of theſe ſeries is $= \frac{S}{3} \times \frac{\beta\varkappa . \overline{FK - AFK}}{bc} - \frac{S}{3r} \times \overline{BC - ABC}$; that the ſecond ſeries is $= \frac{S}{6} \times \frac{\varkappa . \overline{BK - ABK}}{c} - \frac{S}{6r} \times \frac{m . \overline{PC - APC}}{b}$; that the third ſeries is $= \frac{S}{6} \times \frac{\beta \, . \, \overline{FC - AFC}}{b} - \frac{S}{6r} \times \frac{d \, . \, \overline{BT - ABT}}{c}$; and that the fourth ſeries is $= \frac{S}{3} \times \overline{BC - ABC} - \frac{S}{3r} \times \frac{md \, . \, \overline{PT - APT}}{bc}$. Theſe ſeveral expreſſions being added together will be found $=$ S into $\frac{x}{3c} \times \overline{\frac{\beta \, . \, \overline{FK - AFK}}{b} + \frac{BK - ABK}{2}} + \frac{\beta}{6b} \times \overline{FC - AFC} + \frac{\overline{-1}}{3r} \times \overline{BC - ABC} - \frac{m \, . \, \overline{PC - APC}}{6br} - \frac{d}{3cr} \times \overline{\frac{BT - ABT}{2} + \frac{m \, . \, \overline{PT - APT}}{b}}$.

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* See Phil. Tranſ. Vol. LXXIX.

This theorem gives the exact value when either B or C are the oldeſt of the three lives; but when A is the oldeſt, it will be neceſſary to exchange the ſymbols a', a'', a''', &c. for $\overline{a-s}$, $\overline{s-t}$, $\overline{t-u}$, &c. and the ſymbols m, n, o, &c. for $\overline{c-c'}$, $\overline{c-\overline{c'+c''}}$, $\overline{c-\overline{c'+c''+c'''}}$, &c. In this caſe the value of the given ſum for the firſt year will be found $= \frac{S}{abcr}$ into $\frac{abc}{2} - \frac{bsc}{2} + \frac{amc}{2} - \frac{msc}{2} - \frac{abc'}{6} + \frac{bsc'}{6} - \frac{amc'}{3} + \frac{msc'}{3}$; for the ſecond year $= \frac{S}{abcr^2}$ into $\frac{msc}{2} - \frac{mtc}{2} + \frac{nsc}{2} - \frac{ntc}{2} - \frac{msc''}{6} + \frac{mtc''}{6} - \frac{nsc''}{3} + \frac{ntc''}{3} - \frac{msc'}{2} + \frac{mtc'}{2} - \frac{nsc'}{2} + \frac{ntc'}{2}$; for the third year $= \frac{S}{abcr^3}$ into $\frac{ntc}{2} - \frac{nuc}{2} + \frac{otc}{2} + \frac{ouc}{2} - \frac{ntc'''}{6} + \frac{nuc'''}{6} - \frac{otc'''}{3} + \frac{ouc'''}{3} - \frac{nt \, . \, \overline{c'+c''}}{2} + \frac{nu \, . \, \overline{c'+c''}}{2} - \frac{ot \, . \, \overline{c'+c''}}{2} + \frac{ou \, . \, \overline{c'+c''}}{2}$; and ſo on for the remaining years of A's life. Hence the whole value of the given ſum will be $= \frac{S}{2ab} \times \overline{\frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{bs}{r} + \frac{mt}{r^2} + \frac{nu}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{am}{r} + \frac{ns}{r^2} + \frac{ot}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{msc'}{r} + \frac{nt \, . \, \overline{c'+c''}}{r^2} + \&c.} + \frac{S}{6abc} \times \overline{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c.} + \frac{S}{2abcr} \times$

mtc'

$\overline{\frac{mtc'}{r}+\frac{nu\,.\,\overline{c'+c''}}{r^2}+\&c.}-\frac{S}{3abc}\times\overline{\frac{amc'}{r}+\frac{ns\,.\,c''}{r^2}+\frac{ot\,.\,c'''}{r^3}+\&c.}-\frac{S}{2abcr}\times$ $\overline{\frac{nsc'}{r}+\frac{ot\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{3abc}\times\overline{\frac{msc'}{r}+\frac{nt\,.\,c''}{r^2}+\frac{ouc'''}{r^3}+\&c.}\times\frac{S}{2abcr}\times$ $\overline{\frac{nt\,.\,c'}{r}+\frac{ou\,.\,\overline{c'+c''}}{r^2}+\&c.}$ . . . The firſt four ſeries are reſpectively $=\frac{\alpha\beta\,.\,HF}{2ab}$ (or $\frac{1+AB}{2r}$) $-\frac{\beta\,.\,AF}{2b}+\frac{\alpha\,.\,HB}{2a}-\frac{AB}{2}$ . . . The firſt term of the fifth ſeries, or $\frac{abc'}{r}$, is $=\frac{\alpha\beta}{abr}\times\overline{\frac{ab}{\alpha\beta}-\frac{ab\,.\,\overline{c-c'}}{\alpha\beta c}}$, the ſecond term or $\frac{msc''}{r^2}$, is $=\frac{\alpha\beta}{abr^2}\times\overline{\frac{ms}{\alpha\beta}-\frac{ms\,.\,\overline{c-\overline{c'+c''}}}{\alpha\beta c}-\frac{msc'}{r^2}}$ . . . . the third term, or $\frac{ntc'''}{r^3}$, is $=\frac{\alpha\beta}{abr^3}\times\overline{\frac{nt}{\alpha\beta}-\frac{nt\,.\,\overline{c-\overline{c'+c''+c'''}}}{\alpha\beta c}}-\frac{nt'\,.\,\overline{c'+c''}}{r^3}$. Therefore the ſum of the fifth and ſixth ſeries is $=-\frac{\alpha\beta\,.\,\overline{HF-HFC}}{6ab}\Big(-\frac{msc'}{3abcr^2}-\frac{nt\,.\,\overline{c'+c''}}{3abcr^3}-\&c.=\Big)-\frac{1}{3r}\times\overline{AB-ABC}$. In the ſame manner the ſeventh ſeries may be found $=\frac{\beta}{6b}\times\overline{AF-AFC}-\frac{mtc'}{6abcr^2}-\frac{nu\,.\,\overline{c+c''}}{6abcr^3}-\&c.$; and conſequently the ſum of the ſeventh and eighth ſeries is $=\frac{\beta\,.\,\overline{AF-AFC}}{6b}+\frac{s\,.\,\overline{BN-BNC}}{3ar}$. Again, the ninth and tenth ſeries may be found $=-\frac{\alpha\,.\,\overline{HB-HBC}}{3a}\Big(-\frac{nsc'}{6abcr^2}-\frac{ot\,.\,\overline{c'+c''}}{6abcr^3}-\&c.=\Big)-\frac{m\,.\,\overline{AP-APC}}{6br}$; and the eleventh and twelfth or two laſt ſeries, $=\frac{AB-ABC}{3}\Big(+\frac{ntc'}{6abcr^2}+\frac{uo\,.\,\overline{c'+c''}}{6abcr^3}+\&c.=\Big)+\frac{ms\,.\,\overline{PN-PNC}}{6abr}$. If all theſe expreſſions be added together, we ſhall have the value of the given ſum $=$ S into $\frac{\beta}{3b}\times\overline{\frac{\alpha\,.\,\overline{HF+\frac{1}{2}HFC}}{a}-\overline{AF+\frac{1}{2}AFC}}$

$+\frac{1}{6}\times\overline{\frac{\alpha.\overline{HB+2HBC}}{a}-\overline{AB+2ABC}}+\frac{1}{3r}-\overline{\frac{s.\overline{BN-BNC}}{a}-AB-ABC}$ $+\frac{m}{6br}\times\overline{\frac{s.\overline{PN-PNC}}{a}-\overline{AP-APC}}$.

It may be obſerved, that the firſt four ſeries are alſo $=E$, and therefore if this value be ſubſtituted inſtead of $\frac{\alpha\beta.HF}{2ab}$, &c. the general rule will become $=E+S$ into $-\frac{\alpha}{3a}\times\overline{HB-HBC}+\frac{\beta}{2b}\times\overline{HF-HFC}+\frac{\overline{r-1}.\overline{AB-ABC}}{3r}+\frac{\beta.\overline{AF-AFC}}{6b}-\frac{m.\overline{AP-APC}}{6br}+\frac{s}{3ar}\times\overline{BN-BNC}+\frac{2b}{m}\times\overline{PN-PNC}$ …… Suppoſing the three lives to be equal, the firſt of theſe rules will become $=\frac{1+CC}{2r}-\frac{\kappa\kappa.\overline{KK-CKK}}{6.cc}-\frac{1}{3r}\times\overline{CC-CCC}-\frac{\kappa.\overline{CK-CCK}}{6c}+\frac{d.\overline{CT-CCT}}{6cr}-\frac{\overline{CC+2CCC}}{6}+\frac{dd.\overline{TT-CTT}}{6.cc.r}$, and the ſecond $=\frac{\overline{r-1}.\overline{V-CC}}{2r}+\frac{\overline{r-1}.\overline{CC-CCC}}{3r}-\frac{\kappa.\overline{CK-CCK}}{6c}-\frac{\kappa\kappa.\overline{KK-CCK}}{6cc}+\frac{d.\overline{CT-CCT}}{6cr}+\frac{dd.\overline{TT-CTT}}{6ccr}$: and alſo the rule denoting the value (when either B or C are the eldeſt of the three lives) will become $=\frac{\overline{r-1}.\overline{CC-CCC}}{3r}+\frac{\kappa}{3c}\times\overline{CK-CCK}+\frac{\kappa\kappa}{3.cc}\times\overline{KK-CKK}-\frac{d}{c}\times\frac{\overline{CT-CCT}}{3r}-\frac{dd.}{3ccr}\times\overline{TT-CTT}$. Let this laſt expreſſion be called Q and compared with the firſt of the preceding expreſſions. In this caſe we ſhall have $\frac{1+CC}{2r}-\frac{\overline{CC+2CCC}}{6}-\frac{\overline{CC-CCC}}{3r}-\frac{Q}{2}+\frac{\overline{r-1}.\overline{CC-CCC}}{6r}=Q$, from which Q may be eaſily found $=\frac{S}{3}\times\frac{\overline{r-1}.\overline{V-CCC}}{r}$. Again, let the ſame expreſſion, denoted by Q, be compared with the ſecond of the preceding expreſſions, and we ſhall then have $\frac{\overline{r-1}\,.\,\overline{V-CC}}{2r} - \frac{Q}{2} + \frac{\overline{r-1}\,.\,\overline{CC-CCC}}{2r} = Q$, and conſequently $Q = S \times \frac{\overline{r-1}\,.\,\overline{V-CCC}}{3r}$ as before * .... Now it is well known, that when the lives are all equal, the value of the reverſion muſt be *one third* the difference between the perpetuity and the three joint lives, and therefore a demonſtration ariſes of the truth of the whole ſolution. As a ſtill further proof of this, the foregoing theorem may be immediately deduced from the ſeries themſelves: thus, the value of the given ſum for the firſt year will in this caſe be $= \frac{S\,.\,\overline{c^3-d^3}}{3c^3\,.\,r}$; for the ſecond year it will be $= \frac{S\,.\,\overline{d^3-e^3}}{3c^3r^2}$; for the third year it will be $= \frac{S\,.\,\overline{e^3-f^3}}{3c^3r^3}$. Hence the value of the whole reverſion will be $=$

$$\frac{S}{3} \times \overline{\frac{1}{r} - \frac{d^3}{c^3r} - \frac{e^3}{c^3r^2} - \&c.} + \frac{S}{3} \times \overline{\frac{d^3}{c^3r^2} + \frac{e^3}{c^3r^3} + \frac{f^3}{c^3r^4} + \&c.} = \frac{S}{3} \times \overline{\frac{1}{r} - CCC + \frac{CCC}{r}} = \frac{S}{3} \times \frac{\overline{r-1}\,.\,\overline{V-CCC}}{r} \;\ldots\; \text{Q. E. D.}$$

PROBLEM II.

To determine the value of a given ſum, payable if A ſhould be the *ſecond* that fails of the three lives A, B, and C.

SOLUTION.

The ſum S may be received in the firſt year, provided either of three events ſhould happen; 1ſt, if the three lives ſhould become extinct, and A be the ſecond that fails; 2dly, if A ſhould die after B, and C live; and, 3dly, if A ſhould die after C, and B live. But in the ſecond and following years, the given ſum may be received, provided either of ſeven events ſhould happen. 1ſt, If the three lives ſhould fail in the year, A's life having been the ſecond that failed. 2dly, If A only ſhould die in the year, B having died before the beginning and C lived to the end of it. 3dly, If A only ſhould die in the year, C having died before the beginning and B lived to the end of it. 4thly, If A ſhould die after B in the year and C live. 5thly, If A ſhould die after C in the year and B live. 6thly, if A and C ſhould both die in the year (A dying firſt) and B's life ſhould have failed in one or other of the foregoing years; and, 7thly, if A and B ſhould both die in the year (A dying firſt) and C's life ſhould have failed in the foregoing years. From the ſeveral expreſſions denoting theſe contingencies the whole value of the reverſion may be found =

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* This expreſſion may alſo be obtained from either of the above general rules, independent of the two others, in like manner as in the ſolution of the fourth problem.

$$\frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c.} + \frac{S}{2ac} + \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c.} +$$

$$\frac{S}{2ab} \times \overline{\frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a'''n}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c.} -$$

$$\frac{2S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a''ne}{r^3} + \&c.} - \frac{2..S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} -$$

$$\frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.}$$

The two firſt of theſe ſeries are = D, . . the two next = E . . . and it appears from the ſolution of the preceding problem that the four remaining ſeries expreſs double the value of the ſum S, depending on the contingency of A's dying firſt (when B or C are the oldeſt of the three lives) with a negative ſign.

The general rule, therefore, in this caſe will become $= D + E - 2M$.

But when A is the oldeſt life, recourſe muſt be had, as in the ſecond part of the preceding problem, to different ſymbols, and the value of the reverſion will then be found =

$$\frac{S}{ab} \times \overline{-\frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c.} + \frac{S}{ab} \times \overline{\frac{bs}{r} + \frac{mt}{r^2} + \frac{nu}{r^3} + \&c.} +$$

$$\frac{S}{3abc} \times \overline{\frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c.} +$$

$$\frac{2S}{3abc} \times \overline{\frac{amc'}{r} + \frac{nsc''}{r^2} + \frac{otc'''}{r^3} + \&c.} - \frac{2S}{3abc} \times \overline{\frac{msc'}{r} + \frac{ntc''}{r^2} + \frac{ouc'''}{r^3} + \&c.} +$$

$$\frac{S}{abc} \times \overline{\frac{msc'}{r} + \frac{nt\,.\,\overline{c'+c''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{mtc'}{r} + \frac{nu\,.\,\overline{c'+c''}}{r^2} + \&c.} +$$

$$\frac{S}{abc} \times \overline{\frac{nsc'}{r} + \frac{ot\,.\,\overline{c'+c''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{nt\,.\,c'}{r} + \frac{ou\,.\,\overline{c'+c''}}{r^2} + \&c.} +$$

$$\frac{S}{2ab} \times \overline{\frac{\overline{a-s}\,.\,b}{r} + \frac{\overline{s-t}\,.\,m}{r^2} + \frac{\overline{t-u}\,.\,n}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{\overline{a-s}\,.\,m}{r} + \frac{\overline{s-t}\,.\,n}{r^2} + \&c.} +$$

$$\frac{S}{2ac} \times \overline{\frac{\overline{a-s}\,.\,c}{r} + \frac{\overline{s-t}\,.\,\overline{c-c'}}{r^2} + \&c.} + \frac{S}{2ac} \times \overline{\frac{\overline{a-s}\,.\,\overline{c-c'}}{r} + \frac{\overline{s-t}\,.\,\overline{c-c'+c''}}{r^2} +}$$

$$\overline{\&c.} - \frac{S}{ab} \times \overline{\frac{am}{r} + \frac{ns}{r^2} + \frac{ot}{r^3} + \&c.} + \frac{S}{ab} \times \overline{\frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c.}$$

The laſt two, and the firſt ten ſeries are $= -2M$, the eleventh and twelfth ſeries are $= D$, and the thirteenth and fourteenth ſeries $= E$; conſequently the general rule becomes $= D + E - 2M$, as before.

Suppoſing the lives were all equal, the above expreſſion would be $= \frac{S\,.\,\overline{r-1}\,.\,\overline{V-CC}}{r} - \frac{2\,.\,S}{3^r} \times \overline{r-1}\,.\,\overline{V-CCC} = \frac{S}{3} \times \frac{\overline{r-1}}{r} \times \overline{V - \overline{3CC - 2CCC}}$, which is known from other principles * to be the true value, and therefore the inveſtigation is right. As a further demonſtration, however, it may not be improper to

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* See Phil. Tranſ. Vol. LXXIX.

obſerve, that this rule is immediately obtained from the different fractions which expreſs the ſeveral contingencies in each year. For in this caſe, the value of S for the firſt year becomes $= \frac{S}{r} \times \overline{\frac{1}{3} - \frac{dd}{cc} + \frac{2d^3}{c^3}}$, for the ſecond year $= \frac{S}{r^2} \times \overline{\frac{dd}{cc} - \frac{ee}{cc} - \frac{2d^3}{c^3} + \frac{2e^3}{c^3}}$ and ſo on for the other years. Theſe ſeries being added together will be found $= \frac{S}{3} \times \overline{\frac{1}{r} + \frac{2d^3}{c^3 . r} + \frac{2e^3}{c^3 . r^2} + \&c.} - \frac{S}{3} \times \overline{\frac{3dd}{ccr} + \frac{3ee}{cc . r^2} + \&c.} + \frac{S}{3r} \times \overline{\frac{3dd}{ccr} + \frac{3ee}{cc . r^2} + \&c.} - \frac{S}{3r} \times \overline{\frac{2d^3}{c^3 . r^2} + \frac{2e^3}{c^3 . r^2} + \&c.} = \frac{S}{3} \times \overline{\frac{r-1}{r}} \times \overline{V - 3\overline{CC - 2CCC}}$ .... Q. E. D.

PROBLEM III.

To determine the value of a given ſum payable on the death of A, if his life ſhould be the *laſt* that fails of the three lives A, B, C.

SOLUTION.

The given ſum can be received in the firſt year only upon the extinction of the three lives, A having died laſt. In the ſecond and following years it may be received provided either of four events ſhould happen: 1ſt, if all the three lives ſhould fail in the year, A dying laſt; 2dly, if A ſhould die *after* C in the year, B having died in either of the foregoing years; 3dly, if A ſhould die *after* B in the year, C having died in either of the foregoing years; 4thly, if only A ſhould die in the year, B and C having both died before the beginning of it. The value therefore of the reverſion (when B or C are older

than A) will be $= \frac{S}{a}\times\overline{\frac{a'}{r}+\frac{a''}{r^2}+\frac{a'''}{r^3}+\&c.}+\frac{S}{3abc}\times\overline{\frac{a'bc}{r}+\frac{a''md}{r^2}+\frac{a'''ne}{r^3}+\&c.}+\frac{S}{3abc}\times\overline{\frac{a'md}{r}+\frac{a''ne}{r^2}+\frac{a'''of}{r^3}+\&c.}+\frac{S}{6abc}\times\overline{\frac{a'mc}{r}+\frac{a''nd}{r^2}+\frac{a'''oe}{r^3}+\&c.}+\frac{S}{6abc}\times\overline{\frac{a'bd}{r}+\frac{a''me}{r^2}+\frac{a'''nf}{r^3}+\&c.}-\frac{S}{2ac}\times\overline{\frac{a'c}{r}+\frac{a''d}{r^2}+\frac{a'''e}{r^3}+\&c.}-\frac{S}{2ac}\times\overline{\frac{a'd}{r}+\frac{a''e}{r^2}+\frac{a'''f}{r^3}+\&c.}-\frac{S}{2ab}\times\overline{\frac{a'b}{r}+\frac{a''m}{r^2}+\frac{a'''n}{r^3}+\&c.}-\frac{S}{2ab}\times\overline{\frac{a'm}{r}+\frac{a''n}{r^2}+\frac{a'''o}{r^3}+\&c.}=G+M-\overline{D+E}.$

When the life of A is the oldeſt of the three lives, the ſymbols being changed as in the two preceding problems, the value of the reverſion will become $= -\frac{S}{6abc}\times\overline{\frac{abc'}{r}+\frac{msc''}{r^2}+\frac{ntc'''}{r^3}+\&c.}-\frac{S}{2abcr}\times\overline{\frac{msc'}{r}+\frac{nt\,.\,c'+c''}{r^2}+\&c.}+\frac{S}{6abc}\times\overline{\frac{bsc'}{r}+\frac{mtc''}{r^2}+\frac{nuc'''}{r^3}+\&c.}+\frac{S}{2abcr}\times\overline{\frac{mt\,.\,c'}{r}+\frac{nu\,.\,\overline{c'+c''}}{r^2}+\&c.}-\frac{S}{3abc}\times\overline{\frac{amc'}{r}+\frac{ns''c}{r^2}+\frac{ot\,.\,c'''}{r^3}+\&c.}-\frac{S}{2abcr}\times\overline{\frac{ns\,.\,c'}{r}+\frac{ot\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{3abc}\times\overline{\frac{msc'}{r}+\frac{nt\,.\,c''}{r^2}+\frac{ou\,.\,c'''}{r^3}+\&c.}+\frac{S}{2abcr}\times\overline{\frac{nt\,.\,c'}{r}+\frac{ou\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{2ac}\times\overline{\frac{\overline{a-s}\,.\,c'}{r}+\frac{\overline{s-t}\,.\,c''}{r^2}+\frac{\overline{t-u}\,.\,c'''}{r^3}+\&c.}+\frac{S}{acr}\times\overline{\frac{\overline{s-t}\,.\,c'}{r}+\frac{\overline{t-u}\,.\,\overline{c'+c''}}{r^2}+\&c.}$

By the ſecond part of the ſolution of the firſt problem, the firſt eight ſeries may be found $= M - E$; and the laſt two ſeries being eaſily reſolved into $\frac{S}{a}\times\overline{\frac{\overline{a-s}}{r}+\frac{\overline{s-t}}{r^2}+\frac{\overline{t-u}}{r^3}+\&c.}-\frac{S}{2ac}\times\overline{\frac{\overline{a-s}\,.\,\overline{c-c'}}{r}+\frac{\overline{s-t}\,.\,\overline{c-c'+c''}}{r^2}+\&c.}-\frac{S}{2ac}\times\overline{\frac{\overline{a-s}\,.\,c}{r}+\frac{\overline{s-t}\,.\,\overline{c-c'}}{r^2}+\&c.}$ are $= G - D$. Hence the general rule, as in the former caſe, becomes $= G + M - \overline{D+E}$.

When the lives are all equal, the above expreſſion will be changed into $\frac{S}{r} \times \overline{r-1} \times \overline{\overline{V-C} + \frac{V-CCC}{3}} - \overline{V-CC} = \frac{S \, . \, \overline{r-1}}{r} \times \frac{\overline{V-L}}{3}$; or "*one-third* the value of the abſolute reverſion after "the extinction of the longeſt of the three lives." This is known from ſelf-evident principles to be the true value, and therefore the foregoing ſolution is correct. The ſame rule may alſo be obtained immediately from the ſeries; for the value in this caſe for the firſt year will be $= \frac{S}{r} \times \overline{\frac{1}{3} - \frac{d}{c} + \frac{dd}{cc} - \frac{d^3}{3c^3}}$, for the ſecond year $= \frac{S}{r^2} \times \overline{\frac{d^3}{3c^3} - \frac{e^3}{3c^3} - \frac{dd}{cc} + \frac{a}{c} + \frac{ee}{cc} - \frac{e}{c}}$, and ſo on for the remaining years. Hence the whole value of the reverſion will be $= \frac{S}{3r} \times L + \frac{S}{3} \times \overline{\frac{1}{r} - L} = \frac{\overline{r-1} \, . \, S}{r} \times \frac{\overline{V-L}}{3}$ . . . . Q. E. D.

PROBLEM IV.

To determine the value of a given ſum payable on the extinction of the lives of A and B, ſhould they be the *firſt* that fail of the three lives A, B, and C.

SOLUTION.

The given ſum may be received in the firſt year, either on the extinction of the three lives, C having died laſt; or on the extinction of only the two lives A and B, C having ſurvived the year. In the ſecond and following years the given ſum may be received, provided either of ſix events ſhould happen. 1ſt, If the three lives ſhould become extinct in the year, C having been the laſt that failed. 2dly, If A only ſhould die in the year, B having died before the beginning, and C lived to the end of it. 3dly, If B only ſhould die in the year, A having died before the beginning, and C lived to the end of it. 4thly, If A and B ſhould both die in the year, and C ſurvive it. 5thly, If C ſhould die after A in the year, B having died in either of the foregoing years. 6thly, if C ſhould die after B in the year, A having died in either of the foregoing years. The fractions denoting theſe ſeveral contingencies being added together will be found =

$$\frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''.d}{r^2} + \frac{a'''.e}{r^3}} + \&c. + \frac{S}{2ac} \times \overline{\frac{a\,d}{r} + \frac{a''e}{r^2} + \frac{a'''.f}{r^3}} + \&c. -$$

$$\frac{S}{6abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3}} + \&c. + \frac{S}{2abcr} \times \overline{\frac{md\,.\,a'}{r} + \frac{ne\,.\,\overline{a'+a''}}{r^2}} + \&c. -$$

$$\frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3}} + \&c. - \frac{S}{2abcr} \times \overline{\frac{nd\,.\,a'}{r} + \frac{oe\,.\,\overline{a'+a''}}{r^2}} + \&c. +$$

$$\frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''rf}{r^3}} + \&c. + \frac{S}{2abcr} \times \overline{\frac{me\,.\,a'}{r} + \frac{nf\,.\,\overline{a'+a''}}{r^2}} + \&c. -$$

$$\frac{2S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3}} + \&c. - \frac{S}{2abcr} \times \overline{\frac{ne\,.\,a'}{r} + \frac{of\,.\,\overline{a'+a''}}{r^2}} + \&c.$$

The firſt two ſeries are = D; the third and fourth ſeries are = $\frac{2}{3r} \times \overline{BC - ABC} - \frac{\beta\kappa\,.\,\overline{FK - AFK}}{6\,.\,bc}$; the fifth and ſixth ſeries are = $-\frac{\kappa\,.\,\overline{BK - ABK}}{3c} - \frac{m\,.\,\overline{PC - APC}}{6br}$; the ſeventh and eighth ſeries = $\frac{\beta\,.\,\overline{FC - AFC}}{6b} + \frac{d\,.\,\overline{BT - ABT}}{3cr}$; and the ninth and tenth ſeries = $\frac{md\,.\,\overline{PT - APT}}{6bcr} - \frac{2\,.\,\overline{BC - ABC}}{3}$. Hence the whole value of the reverſion will be

$$= D - \frac{S\,.\,\kappa}{3c} \times \overline{\frac{\beta\,.\,\overline{FK - AFK}}{2b} + \overline{BK - ABK}} + \frac{S\,.\,\beta}{6b} \times \overline{FC - AFC} - \frac{2\,.\,S\,.\,\overline{r - 1}}{3r} \times \overline{BC - ABC} - \frac{S\,.\,m}{6br} \times \overline{PC - APC} + \frac{S\,.\,d}{3cr} \times \overline{\overline{BT - ABT} + \frac{m\,.\,\overline{PT - APT}}{2b}}.$$

The preceding theorem expreſſes the value of the given ſum whether C be the oldeſt, or B one of the two lives to be ſurvived, and will therefore be ſufficient in all caſes. In order, however, ſtill more fully to prove this, let A be ſuppoſed the oldeſt life, and inſtead of $\overline{b-m}$, $\overline{m-n}$, $\overline{n-o}$, &c. and a', a'', a''', &c. let b', b'', b''', &c. and $\overline{a-s}$, $\overline{s-t}$, $\overline{t-u}$, &c. be ſubſtituted; then will the value of the reverſion be found $=$

$$\frac{S}{2bc} \times \overline{\frac{cb'}{r} + \frac{db''}{r^2} + \frac{eb'''}{r^3} + \&c.} + \frac{S}{2bc} \times \overline{\frac{ab'}{r} + \frac{eb''}{r^2} + \frac{fb'''}{r^3} + \&c.} -$$

$$\frac{S}{6abc} \times \overline{\frac{acb'}{r} + \frac{dsb''}{r^2} + \frac{etb'''}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{db}{r} + \frac{et\,.\,\overline{b'+b''}}{r^2} + \&c.} -$$

$$\frac{S}{3abc} \times \overline{\frac{csb'}{r} + \frac{dtb''}{r^2} + \frac{eub'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{db'}{r} + \frac{eu\,.\,\overline{b+b''}}{r^2} + \&c.} +$$

$$\frac{S}{6abc} \times \overline{\frac{adb'}{r} + \frac{seb''}{r^2} + \frac{ftb'''}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{seb'}{r} + \frac{tf\,.\,\overline{b'+b''}}{r^2} + \&c.} -$$

$$\frac{2S}{3abc} \times \overline{\frac{dsb'}{r} + \frac{et\,.\,b''}{r^2} + \frac{fu\,.\,b'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{et\,.\,b'}{r} + \frac{fu\,.\,\overline{b'+b''}}{r^2} + \&c.}$$

Let Σ denote the value of S on the contingency of C's ſurviving B, and the general rule deduced from the preceding ſeries will become $= \Sigma - \frac{S\,.\,x}{3} \times \overline{\frac{\alpha\,.\,\overline{HK-HBK}}{2a} + \overline{AK-ABK}} +$ $\frac{S\,.\,x}{6a} \times \overline{HC-HBC} - \frac{2S\,.\,\overline{r-1}}{3r} \times \overline{AC-ABC} - \frac{S\,.\,s}{6ar} \times \overline{NC-NBC} +$ $\frac{S\,.\,d}{3cr} \times \overline{\overline{AT-ABT} + \frac{s\,.\,\overline{NT-NBT}}{2a}}$, which appears to be exactly the ſame rule with the foregoing, if the ſymbols of A and B be only exchanged for each other.

If the three lives be of the ſame age, both thoſe rules will ſeverally become $=$ S into $\frac{\overline{r-1}}{2r} \times \overline{V-CC} - \frac{xx}{6cc} \times \overline{KK-CKK} -$ $\frac{x}{6c} \times \overline{CK-CCK} - \frac{2\,.\,\overline{r-1}}{3r} \times \overline{CC-CCC} + \frac{d}{6cr} \times \overline{CT-CCT} + \frac{dd}{6ccr} \times$

TT –

$\overline{TT-CTT}$. The two expreſſions $-\frac{xx\,.\,\overline{KK-CKK}}{6cc}+\frac{d}{6cr}\times\overline{CT-CCT}$, by reſolving them into their reſpective ſeries, will be found $=\frac{1-CC}{6r}+\frac{d}{6cr}+\overline{\frac{de}{6c^2r^2}+\frac{ef}{6c^2r^3}+\frac{fg}{6c^2r^4}}+$ &c., and the two expreſſions $\frac{dd\,.\,\overline{TT-CTT}}{6ccr}-\frac{x}{c}\times\overline{CK-CCK}=\frac{CC}{6r}-\frac{d}{6cr}-\overline{\frac{de}{6ccr^2}+\frac{ef}{6ccr^3}+\frac{fg}{6ccr^4}+}$ &c.; hence the ſum of theſe four expreſſions will be $=-\frac{\overline{r-1}\,.\,\overline{V-CC}}{6r}$, and conſequently the general rule in this caſe will be $=\frac{S\,.\,\overline{r-1}}{3r}\times\overline{V-\overline{3CC-2CCC}}$, which is known to denote the true value from other principles *.

As a further proof of the accuracy of the preceding inveſtigation, it may not be improper to obſerve, that this rule may be immediately obtained from the different fractions which expreſs the value of the given ſum in each year. For in this caſe the value of S in the firſt year is $=\frac{S}{3}\times\overline{\frac{1}{r}-\frac{3dd}{ccr}+\frac{2d^3}{c^3\,.\,r}}$, in the ſecond year $=\frac{S}{3}\times\overline{\frac{2e^3}{c^3r^2}-\frac{3ee}{ccr^2}-\frac{2d^3}{c^3r^2}+\frac{3dd}{ccr^2}}$, and ſo on in the other years, which expreſſions may be eaſily found $=\frac{S\,.\,\overline{r-1}}{3r}\times\overline{V-\overline{3CC-2CCC}}$. Q. E. D.

PROBLEM V.

To find the value of a given ſum, payable on the death of A, if his life ſhould be the *firſt* or *ſecond* that fails of the three lives A, B, and C.

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* See my Paper in the Phil. Tranſ. Vol. LXXIX.

SOLUTION.

In the firſt year the payment of the given ſum depends upon either of four events. 1ſt, That all the three lives ſhall become extinct, the life of A having been the firſt or ſecond that failed. 2dly, That A and B ſhall both die, and C live to the end of the year. 3dly, That A and C ſhall both die, and B live to the end of the year. 4thly, That only A ſhall die, and B and C both live to the end of the year. In the ſecond and following years the payment of the given ſum will depend upon either of eight events happening. 1ſt, That all the three lives ſhall drop in the year, A having been the firſt or ſecond that failed. 2dly, That C ſurvives, and only A and B die in the year. 3dly, That B ſurvives, and only A and C die in the year. 4thly, That both B and C ſurvive, and A only dies in the year. 5thly, That A dies in the year, B having died before the beginning, and C lived to the end of it. 6thly, That A in like manner dies in the year, C having died before the beginning, and B lived to the end of it. 7thly, That C dies after A in the year, B's life having failed in either of the preceding years. 8thly, That B dies after A in the year, C's life having failed in either of the preceding years. The fractions denoting theſe ſeveral contingencies are

$$\begin{aligned}
&= \frac{S}{2ac}\times\overline{\frac{a'c}{r}+\frac{a''.d}{r^2}+\frac{a'''.e}{r^3}+\&c.} + \\
&\frac{S}{2ac}\times\overline{\frac{a'd}{r}+\frac{a''.e}{r^2}+\frac{a'''f}{r^3}+\&c.} + \frac{S}{2ab}\times\overline{\frac{a'b}{r}+\frac{a''.m}{r^2}+\frac{a'''n}{r^3}+\&c.} + \\
&\frac{S}{2ab}\times\overline{\frac{a'm}{r}+\frac{a''n}{r^2}+\frac{a'''o}{r^3}+\&c.} - \frac{S}{3abc}\times\overline{\frac{a'bc}{r}+\frac{a''md}{r^2}+\frac{a'''ne}{r^3}+\&c.} - \\
&\frac{S}{6abc}\times\overline{\frac{a'm}{r}+\frac{a''nd}{r^2}+\frac{a''oe}{r^3}+\&c.} - \frac{S}{6abc}\times\overline{\frac{a'bd}{r}+\frac{a''me}{r^2}+\frac{a'''nf}{r^3}+\&c.} - \\
&\frac{S}{3abc}\times\overline{\frac{a'md}{r}+\frac{a'n}{r^2}+\frac{a'''ot}{r^3}+\&c.} = D+E-M.
\end{aligned}$$

This general rule gives the true value whether the life of A be older or younger than both or either of the lives of B and C. When the three lives are of equal age, the value of S for the firſt year will be $= \frac{S}{r}\times\overline{\frac{2}{3}+\frac{d^3}{c^3}-\frac{dd}{cc}}$, for the ſecond year $= \frac{S}{r^2}\times\overline{\frac{e^3}{3c^3}-\frac{ee}{cc}-\frac{d^3}{3c^3}+\frac{dd}{cc}}$, and ſo on for the other years. Hence the whole value in this caſe will be $= \frac{S\cdot\overline{r-1}}{3r}\times\overline{2V-\overline{3CC-CCC}}$, which expreſſion may alſo be derived from the general rule juſt given above, or $D+E-M$.

The ſolution of this problem may alſo be obtained either from the firſt and ſecond, or from the third problems. In the one caſe the value of S is evidently equal to the *ſum* of the two values determined by the two firſt-mentioned problems, or $D+E-2M+M=D+E-M$. And in the other caſe its value is equal to the *difference* between the abſolute value of the reverſion after A ($=G$) and its value depending upon the contingency that A ſhall be the *laſt* life that ſhall fail, which being $=G+M-\overline{D+E}$ by the third problem, it follows, that the general rule on this ſuppoſition will be alſo $=D+E-M$. Q. E. D.

PROBLEM VI.

To find the value of a given ſum payable on the death of A, ſhould his life be the *ſecond* or *third* that ſhall fail of the three lives A, B, and C.

SOLUTION.

The payment of the given ſum in the firſt year will depend upon the contingency of either of three events. 1ſt, That all the three lives ſhall become extinct, A having been the ſecond or third that has failed. 2dly, That A ſhall die after B, and C live to the end of the year. 3dly, That A ſhall die after C, and B live to the end of the year. In the ſecond and following years the given ſum will become payable, provided either of eight events ſhould happen. 1ſt, If all the three lives ſhould fail in the year, A having been the ſecond or third that died. 2dly, If A ſhould die after B in the year, and C live to the end of it. 3dly, If A ſhould die after C in the year, and B live to the end of it. 4thly, If A and C ſhould both die in the year, B having died before the beginning of it. 5thly, If A and B ſhould both die in the year, C having died before the beginning of it. 6thly, If A only ſhould die in the year, the lives of B and C having become extinct in either of the preceding years. 7thly, If A ſhould die in the year, B having died before the beginning, and C lived to the end of it. 8thly, if A ſhould die in the year, C having died before the beginning, and B lived to the end of it. Hence the whole value of the reverſion will be found $=$ $\frac{S}{a} \times \overline{\frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} = G - M.$

This rule is correct in all caſes; but when the three lives are equal it becomes more ſimple, and is $= \frac{S \cdot \overline{r-1}}{3r} \times \overline{2V - \overline{3C - CCC}}$; which expreſſion may likewiſe be obtained immediately from the ſeries given above.

The ſolution of this problem, like that of the foregoing one, may alſo be derived from the firſt three problems; for the value of S is either equal to the *difference* between the abſolute value of the reverſion after the death of A and its value depending on the contingency that A ſhall be the *firſt* that fails (found by Prob. 1.), or it is equal to the *ſum* of the two values depending on the contingencies that A ſhall be the *ſecond* or *third* that fails (found by Prob. 2. and 3.). In both caſes the general rule is $= G - M$. Q. E. D.

PROBLEM VII.

To find the value of a given ſum payable on the death of A, ſhould his life be the *firſt* or *laſt* that fails of the three lives A, B, C.

SOLUTION.

In order to receive the given ſum in the firſt year, it is neceſſary that either of four events ſhould happen. 1ſt, That all the three lives ſhould fail, A having been the firſt or third that died. 2dly, That B ſhould die after A, and C live. 3dly, That C ſhould die after A, and B live. 4thly, That A only ſhould die, and B and C both live. In the ſecond and following years the given ſum may be received, provided either of ſeven events ſhould happen. 1ſt, If the three lives ſhould fail, A having been the firſt or laſt that died. 2dly, if B ſhould die after A in the year, and C live to the end of it. 3dly, If C ſhould die after A in the year, and B live to the end of it. 4thly, If A only ſhould die in the year, and B and C both live to the end of it. 5thly, If A's life ſhould fail after that of B in the year, C's life having failed before the beginning of it. 6thly, If A ſhould fail after C in the year, B having failed before the beginning of it. 7thly, If A only ſhould die in the year, B and C having died in either of the preceding years. From the fractions denoting theſe ſeveral contingencies the whole value of the reverſion will be found $=$

$$\frac{S}{a} \times \overline{\frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c.} - \frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c.} -$$
$$\frac{S}{2ac} \times \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a''n}{r^3} + \&c.} -$$
$$\frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c.} + \frac{2S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} +$$
$$\frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} +$$
$$\frac{2S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} = G - \overline{D+E} + 2M.$$

This general theorem will give the exact value in all caſes; but when the lives are equal, it is rendered more ſimple, by ſubſtituting the ſeveral values of G, D, E, and M, and will then become $= \frac{S \,.\, \overline{r-1}}{3r} \times \overline{2V - 3C - 3CC + 2CCC}$; which expreſſion may alſo be derived in this particular caſe from the different ſeries denoting the value of S in each year.

The ſolution of this, like thoſe of the two preceding problems, may likewiſe be obtained by the aſſiſtance of the firſt three problems. For the value of this contingent reverſion is either equal to the *ſum* of the two values of S payable on the death of A, if his life ſhould be the firſt, or if it ſhould be the laſt that fails (found by Prob. 1. and 3.), or it is equal to the *difference* between the value of the abſolute reverſion after A's death, and the value of the contingent reverſion after A's death, provided he ſhould be the *ſecond* that fails of the three lives (found by Prob. 2.). In both caſes the general rule becomes $= G - \overline{D+E} + 2M$. Q. E. D.

PROBLEM VIII.

To find the value of a given ſum payable on the death of A or B, ſhould *either* of them be the *firſt* that ſhall fail of the three lives A, B, and C.

SOLUTION.

In each year the payment of the given ſum will depend upon either of ſix events. 1ſt, If the three lives ſhould fail in the year, A or B having died firſt. 2dly, if A and B ſhould die in the year, and A live. 3dly, If C ſhould die after A in the year, and B live. 4thly, If C ſhould die after B in the year, and A live. 5thly, If A only ſhould die in the year, and B and C both live. 6thly, If B only ſhould die in the year, and A and C both live. The fractions denoting theſe ſeveral contingencies are $= \frac{2S}{3abc} \times \overline{\frac{bca'}{r} + \frac{md \cdot a''}{r^2} + \frac{nea'''}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{mca'}{r} + \frac{nda''}{r^2} + \frac{oea'''}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{bda'}{r} + \frac{mea''}{r^2} + \frac{nf \,.\, a'''}{r^3} + \&c.} + \frac{S}{6abc} \times \overline{\frac{mda'}{a} + \frac{nea''}{r^2} + \frac{ofa'''}{r^3} + \&c.} + \frac{S}{2bc} \times \overline{\frac{\overline{b-m} \,.\, c}{r} + \frac{\overline{m-n} \,.\, d}{r^2} + \frac{\overline{n-o} \,.\, e}{r^3} + \&c.} + \frac{S}{2bc} \times \overline{\frac{\overline{b-m} \,.\, d}{r} + \frac{\overline{m-n} \,.\, e}{r^2} + \frac{\overline{n-o} \,.\, f}{r^3} + \&c.} - \frac{S}{2abc} \times \overline{\frac{bca'}{r} + \frac{md \,.\, \overline{a'+a''}}{r^2} + \&c.} + \frac{S}{2abc} \times \overline{\frac{mca'}{r} + \frac{nd \,.\, \overline{a'+a'}}{r^2} + \&c.} - \frac{S}{2abc} \times \overline{\frac{bda'}{r} + \frac{me \,.\, \overline{a'+a''}}{r^2} + \&c.} + \frac{S}{2abc} \times \overline{\frac{md \,.\, a'}{r} + \frac{ne \,.\, \overline{a'+a''}}{r^2} + \&c.} = S$ into $\frac{\varkappa}{3c} \times \overline{\frac{\beta \,.\, \overline{FK-AFK}}{2b} + BK - ABK} - \frac{\beta \,.\, \overline{FC-AFC}}{6b} + \frac{2 \,.\, \overline{r-1} \,.\, \overline{BC-ABC}}{3r} + \frac{m \,.\, \overline{PC-APC}}{6br} - \frac{d}{3cr} \times \overline{BT - ABT + \frac{m \,.\, \overline{PT-APT}}{2b}} + \Sigma$ (Σ denoting the value of S on the contingency of C's ſurviving B, as in Prob. 4.)

The above rule gives the exact value when C is the oldeſt of the three lives. But if A be the oldeſt, the ſymbols muſt be changed as in ſome of the foregoing problems, and the value in this caſe will be expreſſed by the ſeries $\frac{2S}{3abc}\times\overline{\frac{abc'}{r}+\frac{msc''}{r^2}+\frac{ntc'''}{r^3}+\&c.}-\frac{S}{6abc}\times\overline{\frac{amc'}{r}+\frac{nsc''}{r^2}+\frac{otc'''}{r^3}+\&c.}-\frac{S}{6abc}\times\overline{\frac{bsc'}{r}+\frac{mtc''}{r^2}+\frac{nuc'''}{r^3}+\&c.}-\frac{S}{3abc}\times\overline{\frac{msc'}{r}+\frac{ntc''}{r^2}+\frac{ouc'''}{r^3}+\&c.}+\frac{S}{ab}\times\overline{\frac{ab}{r}+\frac{ms}{r^2}+\frac{nt}{r^3}+\&c.}-\frac{S}{ab}\times\overline{\frac{ms}{r}+\frac{nt}{r^2}+\frac{ou}{r^3}+\&c.}-\frac{S}{abc}\times\overline{\frac{abc'}{r}+\frac{ms\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{abc}\times\overline{\frac{msc'}{r}+\frac{nt\,.\,\overline{c'+c''}}{r^2}+\&c.}$ From theſe ſeveral ſeries, the general rule expreſſing the value of the reverſion will be found $=$ S into $\frac{\overline{r-1}\,.\,\overline{V-AB}}{r}-\frac{\alpha}{3a}\times\overline{\frac{\beta\,.\,\overline{HF-HFC}}{b}+\frac{HB-HBC}{2}}-\frac{\beta\,.\,\overline{AF-AFC}}{6b}+\frac{2\,.\,\overline{r-1}\,.\,\overline{AB-ABC}}{3r}+\frac{m\,.\,\overline{AP-APC}}{6br}+\frac{S}{3ar}\times\overline{\frac{BN-BNC}{2}+\frac{m\,.\,\overline{PN-PNC}}{b}}$.

When the lives are all equal, the firſt rule becomes $=\frac{\overline{r-1}\,.\,\overline{V-CC}}{2r}+\frac{2\,.\,\overline{r-1}\,.\,\overline{CC-CCC}}{3r}+\frac{\varkappa\varkappa\,.\,\overline{KK-CKK}}{6cc}+\frac{\varkappa\,.\,\overline{CK-CCK}}{6c}-\frac{d\,.\,\overline{CT-CCT}}{6cr}-\frac{dd\,.\,\overline{TT-CTT}}{6ccr}$, and the ſecond rule $=\frac{\overline{r-1}\,.\,\overline{V-CC}}{r}+\frac{2\,.\,\overline{r-1}\,.\,\overline{CC-CCC}}{3r}-\frac{\varkappa\varkappa\,.\,\overline{KK-CKK}}{3cc}-\frac{\varkappa\,.\,\overline{KC-CCK}}{3c}+\frac{d\,.\,\overline{CT-CCT}}{3cr}+\frac{dd\,.\,\overline{TT-CTT}}{3ccr}$. In the one caſe the four laſt fractions are $=\frac{\overline{r-1}\,.\,\overline{V-CC}}{6r}$; and in the other caſe thoſe fractions are $=-\frac{\overline{r-1}\,.\,\overline{V-CC}}{3r}$; therefore, in both caſes the general rule becomes $=\frac{2\,.\,S\,.\,\overline{r-1}}{3r}\times\overline{V-CCC}$, which is known to be the true value from ſelf-evident principles. This expreſſion may alſo be immediately derived from the different fractions which denote the value of S in each year. For in the firſt year theſe fractions will be reduced to $\frac{2S}{3} \times \overline{\frac{1}{r} - \frac{d^3}{c^3 r}}$, in the ſecond year to $\frac{2S}{3} \times \overline{\frac{d^3 - e^3}{c^3 r^2}}$, in the third year to $\frac{2S}{3} \times \overline{\frac{e^3}{c^3 r^3} - \frac{f^3}{c^3 r^3}}$, and ſo on in the other years. Hence the whole value of the reverſion will, as above, be $= \frac{2 . S . \overline{r - 1}}{3r} \times \overline{V - CCC}$. Q. E. D *.

PROBLEM IX.

To determine the value of a given ſum payable on the death of A or B, ſhould *either* of them be the *ſecond* that fails of the three lives A, B, and C.

SOLUTION.

The payment of the given ſum in the firſt year will depend upon either of four events happening. 1ſt, That the three lives ſhould fail, A or B having been the ſecond that failed. 2dly, That A ſhould die after C, and B live. 3dly, That B ſhould die after C, and A live. 4thly, That A and B ſhould both die, and C live. In the ſecond and following years the given ſum will become payable, provided either of eleven events ſhould happen. 1ſt, If the three lives ſhould drop in the year, A or B having been the ſecond that failed. 2dly, If A ſhould die after C in the year, and B live. 3dly, If B ſhould die after C in the year, and A live. 4thly, If A and B ſhould both die in the year, and C live. 5thly, if B only ſhould die in the year, A having died before the beginning, and C lived to the end of it. 6thly, If A only ſhould die in the year, B having died before the beginning, and C lived to the end of it. 7thly, If C ſhould die after A in the year, B having died in either of the foregoing years. 8thly, If C ſhould die after B in the year, A having died in either of the foregoing years. 9thly, If A and B ſhould both die in the year, C having died in either of the preceding years. 10thly, If B only ſhould die in the year, C having died before the beginning, and A lived to the end of it. And, laſtly, if A only ſhould die in the year, C having died before the beginning, and B lived to the end of it. The ſeveral fractions denoting theſe contingencies in each year being added together will become

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* I do not know that any ſolution has been attempted before, either of this or of the two following problems.

$$= \frac{S}{abc} \times \overline{\frac{bca'}{r} + \frac{md\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{mca'}{r} + \frac{nd\,.\,\overline{a'+a''}}{r^2} + \&c.}$$

$$+ \frac{S}{abc} \times \overline{\frac{bda'}{r} + \frac{me\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{mda'}{r} + \frac{ne\,.\,\overline{a'+a''}}{r^2} + \&c.} -$$

$$\frac{4S}{3abc} \times \overline{\frac{bca'}{r} + \frac{mda''}{r^2} + \frac{nea'''}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{mca'}{r} + \frac{nda''}{r^2} + \frac{oea''}{r^3} + \&c.} -$$

$$\frac{2S}{3abc} \times \overline{\frac{bda'}{r} + \frac{mea''}{r^2} + \frac{nfa'''}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{mda'}{r} + \frac{nea''}{r^2} + \frac{of\,.\,a'''}{r^3} + \&c.} -$$

$$\frac{S}{2bc} \times \overline{\frac{\overline{b-m}\,.\,c}{r} + \frac{\overline{m-n}\,.\,d}{r^2} + \frac{\overline{n-o}\,.\,e}{r^3} + \&c.} - \frac{S}{2bc} \times \overline{\frac{\overline{b-m}\,.\,d}{r} + \frac{\overline{m-n}\,.\,e}{r^2} + \frac{\overline{n-o}\,.\,f}{r^3}}$$

$$\overline{+ \&c.} + \frac{S}{ab} \times \overline{\frac{ma'}{r} + \frac{n\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{ab} \times \overline{\frac{ba'}{r} + \frac{m\,.\,\overline{a'+a''}}{r^2} + \&c.} +$$

$$\frac{S}{ab} \times \overline{\frac{ba'}{r} + \frac{ma''}{r^2} + \frac{na''}{r^3} + \&c.} + \frac{S}{b} \times \overline{\frac{\overline{b-m}\,.}{r} + \frac{\overline{m-n}\,.}{r^2} + \frac{\overline{n-o}}{r^3} + \&c.} +$$

$$\frac{S}{2ac} \times \overline{\frac{ca'}{r} + \frac{da''}{r^2} + \frac{ea'''}{r^3} + \&c.} + \frac{S}{2ac} \times \overline{\frac{da'}{r} + \frac{ea''}{r^2} + \frac{fa'''}{r^3} + \&c.}$$

Let Q denote the value of S by the firſt rule in the eighth problem, and the firſt ten ſeries will be $= \Sigma - 2Q$. The four next ſeries are $=\frac{S\,.\,\overline{r-1}\,.\,\overline{V-AB}}{r}$; and the two laſt ſeries are $=D$; the whole value of the reverſion, therefore (when C is the oldeſt of the three lives), is $=\frac{S\,.\,\overline{r-1}\,.\,\overline{V-AB}}{r}+D+\Sigma-2Q.$

When A is the oldeſt of the three lives, the different ſeries, by changing the ſymbols, as in ſome of the foregoing problems, will become

$$=\frac{2S}{abc}\times\overline{\frac{abc'}{r}+\frac{ms\,.\,\overline{c'+c''}}{r^2}+\&c.}-\frac{2S}{abc}\times\overline{\frac{msc'}{r}+\frac{nt\,.\,\overline{c'+c''}}{r^2}+\&c.}-$$

$$\frac{4S}{3abc}\times\overline{\frac{abc'}{r}+\frac{msc''}{r^2}+\frac{nt\,.\,c'''}{r^3}+\&c.}+\frac{2S}{3abc}\times\overline{\frac{ms\,.\,c'}{r}+\frac{nt\,.\,c''}{r^2}+\frac{ou\,.\,c'''}{r^3}+\&c.}$$

$$+\frac{S}{3abc}\times\overline{\frac{amc'}{r}+\frac{nsc''}{r^2}+\frac{otc'''}{r^3}+\&c.}+\frac{S}{3abc}\times\overline{\frac{bsc'}{r}+\frac{mtc''}{r^2}+\frac{nuc'''}{r^3}+\&c.}-$$

$$\frac{S}{bc}\times\overline{\frac{bc'}{r}+\frac{m\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{bc}\times\overline{\frac{mc'}{r}+\frac{n\,.\,\overline{c'+c''}}{r^2}+\&c.}+$$

$$\frac{S}{2abc}\times\overline{\frac{bc'}{r}+\frac{mc''}{r^2}+\frac{nc'''}{r^3}+\&c.}-\frac{S}{2bc}\times\overline{\frac{mc'}{r}+\frac{nc''}{r^2}+\frac{oc''}{r^3}+\&c.}-$$

$$\frac{S}{ac}\times\overline{\frac{ac'}{r}+\frac{s\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{ac}\times\overline{\frac{sc'}{r}+\frac{t\,.\,\overline{c'+c''}}{r^2}+\&c.}+$$

$$\frac{S}{2ac}\times\overline{\frac{ac'}{r}+\frac{sc''}{r^2}+\frac{tc'''}{r^3}+\&c.}-\frac{S}{2ac}\times\overline{\frac{sc'}{r}+\frac{tc''}{r^2}+\frac{uc'''}{r^3}+\&c.}+$$

$$\frac{S}{b}\times\overline{\frac{\overline{b-m}}{r}+\frac{\overline{m-n}}{r^2}+\frac{\overline{n-o}}{r^3}+\&c.}-\frac{S}{ab}\times\overline{\frac{\overline{b-m}\,.\,s}{r}+\frac{\overline{m-n}\,.\,t}{r^2}+\frac{\overline{n-o}\,.\,u}{r^3}+\&c.}$$

$$+\frac{S}{abc}\times\overline{\frac{\overline{b-m}\,.\,s}{r}+\frac{\overline{m-n}\,.\,t}{r^2}+\&c.}$$

Let Q repreſent the value of S by the ſecond rule in the eighth problem, Π and Δ the values of the ſame ſum on the contingency of B's dying after C, and on the contingency of A's dying after C reſpectively *; then will the general rule in this caſe become $=\frac{S\,.\,\overline{r-1}\,.\,\overline{3V-B-A-AB}}{r}-2Q-\overline{\Pi+\Delta}$, or, becauſe $D+\Sigma$ are $=\frac{\overline{r-1}\,.\,\overline{2V-B-A}}{r}-\overline{\Pi+\Delta}$ †, it

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* See my 3d Prob. in Phil. Tranſ. Vol. LXXVIII.

† See my 2d and 3d Prob. Vol. LXXVIII.

it will be $= \frac{S\,.\,\overline{r-1}\,.\,\overline{V-AB}}{r} + D + \Sigma - 2Q$, as above.

When the lives are equal, the rule may be found either immediately from the ſeries, or from the foregoing expreſſions, $=$

$$\frac{2S\,.\,\overline{r-1}}{3r} \times \overline{V - 3CC - 2CCC}. \quad \text{Q. E. D.}$$

PROBLEM X.

To find the value of a given ſum payable on the deceaſe of B or C, ſhould *either* of them be the *laſt* that fails of the three lives A, B, and C.

SOLUTION.

The ſum S can be received in the firſt year only on the extinction of the three lives, reſtrained to the contingency of A's life having been the firſt or ſecond that failed. In the ſecond and following years it may be received provided either of ſix events ſhould happen. 1ſt, If the three lives ſhould fail in the year, A having been the firſt or ſecond that died. 2dly, If B and C ſhould both die in the year, A having died before the beginning of it. 3dly, If C only ſhould die in the year, A and B having died in either of the preceding years. 4thly, If B only ſhould die in the year, A and C having died in either of the preceding years. 5thly, If B ſhould die after A in the year, C having died before the beginning of it. 6thly, If C ſhould die after A in the year, B having died before the beginning of it. From the fractions expreſſing theſe ſeveral contingencies, the whole value of the reverſion will be found $=$

$$\frac{2S}{3abc} \times \overline{\frac{mda'}{r} + \frac{ne\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{3abc} \times \overline{\frac{bca'}{r} + \frac{md\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{6abc}$$

$$\frac{S}{6abc}\times\overline{\frac{mca'}{r}+\frac{nda''}{r^2}+\frac{oea'''}{r^3}+\&c.}-\frac{S}{6abc}\times\overline{\frac{bda'}{r}+\frac{mea''}{r^2}+\frac{nf\,.\,a'''}{r^3}+\&c.}+$$

$$\frac{S}{2ab}\times\overline{\frac{ba'}{r}+\frac{m\,.\,\overline{a'+a''}}{r^2}+\frac{n\,.\,\overline{a'+a''+a'''}}{r^3}+\&c.}-\frac{S}{2ab}\times\overline{\frac{ma'}{r}+\frac{n\,.\,\overline{a'+a''}}{r^2}+\&c.}+$$

$$\frac{S}{2ac}\times\overline{\frac{ca'}{r}+\frac{d\,.\,\overline{a'+a''}}{r^2}+\&c.}-\frac{S}{2ac}\times\overline{\frac{da'}{r}+\frac{e\,.\,\overline{a'+a''}}{r^2}+\&c.}-$$

$$\frac{2S}{3abcr}\times\overline{\frac{mda'}{r}+\frac{ne\,.\,\overline{a'+a''}}{r^2}+\&c.}+\frac{S}{3abcr}\times\overline{\frac{nea'}{r}+\frac{of\,.\,\overline{a'+a''}}{r^2}+\&c.}+$$

$$\frac{S}{2abr}\times\overline{\frac{ma'}{r}+\frac{n\,.\,\overline{a'+a''}}{r^2}+\&c.}-\frac{S}{2abr}\times\overline{\frac{na'}{r}+\frac{o\,.\,\overline{a'+a''}}{r^2}+\&c.}+$$

$\frac{S}{2acr}\times\overline{\frac{d\,.\,a'}{r}+\frac{e\,.\,\overline{a'+a''}}{r^2}+\&c.}-\frac{S}{2acr}\times\overline{\frac{ea'}{r}+\frac{f\,.\,\overline{a'+a''}}{r^2}+\&c.}$, and therefore the general rule (when either B or C are the oldeſt of the three lives) will be $=$ S into $\frac{2\,.\,\overline{r-1}\,.\,\overline{BC-ABC}}{3r}-\frac{\varkappa}{3c}\times$

$$\overline{\frac{\beta\,.\,\overline{FK-AFK}}{b}+\frac{\overline{BK-ABK}}{2}}\times\frac{m}{3br}\times\overline{\frac{\overline{PC-APC}}{2}+\frac{d\,.\,\overline{PT-APT}}{c}}-\frac{\beta\,.\,\overline{FC-AFC}}{6b}$$

$$+\frac{d\,\overline{BT-ABT}}{6cr}+\frac{\beta\,.\,\overline{F-AF}}{2b}-\frac{\overline{r-1}\,.\,\overline{B-AB}}{2r}-\frac{m\,.\,\overline{P-AP}}{2br}+\frac{\varkappa\,.\,\overline{K-AK}}{2c}-$$

$\frac{\overline{r-1}\,.\,\overline{C-AC}}{2r}-\frac{d\,.\,\overline{T-AT}}{2cr}$; or (ſince the firſt five expreſſions are $=\frac{\overline{r-1}\,.\,\overline{BC-ABC}}{r}-$M, and the ſix remaining ones $=$ R $+$ W) it will be $=\frac{S\,.\,\overline{r-1}\,.\,\overline{BC-ABC}}{r}+R+W-M$.

When A *is the oldeſt of the three lives*, the ſame general rule may be obtained. In this caſe, by exchanging the ſymbols, the different ſeries will become $=\frac{S}{6abc}\times\overline{\frac{abc'}{r}+\frac{ms\,.\,\overline{c'+c''}}{r^2}+\&c.}-$

$$\frac{S}{6abc}\times\overline{\frac{bs\,.\,c'}{r}+\frac{mt\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{3abc}\times\overline{\frac{amc'}{r}+\frac{ns\,.\,\overline{c'+c''}}{r^2}+\&c.}+$$

$$\frac{2S}{3abc}\times\overline{\frac{ms\,.\,c'}{r}+\frac{nt\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{c}\times\overline{\frac{c'}{r}+\frac{c''}{r^2}+\frac{c'''}{r^3}+\&c.}-$$

$$\frac{S}{bc}\times\overline{\frac{mc'}{r}+\frac{n\,.\,\overline{c'+c'}}{r^2}}+\&c.-\frac{S}{2ac}\times\overline{\frac{ac'}{r}+\frac{sc''}{r^2}+\frac{tc'''}{r^3}}+\&c.-$$

$$\frac{S}{2ac}\times\overline{\frac{sc'}{r}+\frac{t\,.\,c''}{r^2}+\frac{uc'''}{r^3}}+\&c.-\frac{2S}{3abcr}\times\overline{\frac{ms\,.\,c'}{r}+\frac{nt\,.\,\overline{c+c''}}{r^2}}+\&c.-$$

$$\frac{S}{3abcr}\times\overline{\frac{mt\,.\,c'}{r}+\frac{ru\,.\,\overline{c'+c''}}{r^2}}+\&c.+\frac{S}{6abcr}\times\overline{\frac{rsc'}{r}+\frac{ot\,.\,\overline{c'+c''}}{r^2}}+\&c.-$$

$$\frac{S}{6abcr}\times\overline{\frac{nt\,.\,c'}{r}+\frac{ou\,.\,\overline{c'+c''}}{r^2}}+\&c.+\frac{S}{bcr}\times\overline{\frac{mc'}{r}+\frac{n\,.\,\overline{c'+c''}}{r^2}}+\&c.=\text{S into}$$

$$\frac{2\,.\,\overline{r-1}\,.\,\overline{AB-ABC}}{3r}+\frac{\beta}{6b}\times\overline{\frac{\alpha\,.\,\overline{HF-HFC}}{a}-\overline{AF-AFC}}+\frac{\alpha\,.\,\overline{HB-HBC}}{3a}$$

$$-\frac{s}{3ar}\times\overline{\overline{NB-NBC}+\frac{m\,.\,\overline{PN-PNC}}{2b}}+\frac{m\,.\,\overline{AP-APC}}{6br}+\frac{\overline{r-1}\,.\,\overline{V-C}}{r}-$$

$\frac{\overline{r-1}\,.\,\overline{B-BC}}{r}-\frac{\overline{r-1}\,.\,\overline{A-AC}}{2r}-\frac{\alpha\,.\,\overline{H-HC}}{2a}+\frac{s\,.\,\overline{N-NC}}{2ar}$. In order to get the ſame general rule as that given above in the caſe of B or C's being the oldeſt of the three lives, it is to be obſerved, that the firſt five expreſſions, by the ſecond part of the ſolution of Prob. 1. appear to be $=\frac{\overline{r-1}\,.\,\overline{AB-ABC}}{r}-M+E$. And (ſuppoſing Γ to repreſent the value of S on the contingency of A's dying after C) that the laſt three expreſſions appear to be $=-\Gamma-\frac{\overline{r-1}\,.\,\overline{A-AC}}{r}$. But E is $=R+\frac{\overline{r-1}\,.\,\overline{B-AB}}{r}$, and W muſt be $=\frac{\overline{r-1}\,.\,\overline{V-A-C+AC}}{r}-\Gamma$; therefore, the ſum of the above expreſſions may be eaſily found $=\frac{S\,.\,\overline{r-1}\,.\,\overline{BC-ABC}}{r}+R+W-M$.

If the three lives be of equal age, the value of the reverſion will be $=\frac{2S\,.\,\overline{r-1}}{3r}\times\overline{V-L}$. This expreſſion may be derived either from the foregoing general rule, or immediately from the different ſeries, and is known to be accurately true from ſelf-evident principles. Q. E. D.

I have now given general rules for determining the values of reverſions depending upon three lives in every caſe which, as far as I can diſcover, will admit of an *exact* ſolution. The remaining caſes, which are nearly equal in number to thoſe I have inveſtigated, involve a contingency for which it appears very difficult to find ſuch a general expreſſion as ſhall not render the rules much too complicated and laborious. The contingency to which I refer is that of *one life's failing after another in any given time.* The fractions expreſſing this probability are every year increaſing, ſo that the value of the reverſion muſt be repreſented by as many ſeries at leaſt as are equal to the difference between the age of one of the lives, and that of the oldeſt life in the table of obſervations. I have indeed ſo far ſucceeded in the method of approximation as that the reverſion may be generally aſcertained within about $\frac{1}{50}$th part of its exact value; but I ſhall not trouble the Royal Society at preſent with theſe inveſtigations.

The 34th, 35th, and 36th problems in Mr. SIMPSON's Select Exerciſes involve this contingency, and, by the aſſiſtance of M. DE MOIVRE's hypotheſis, admit of an eaſy ſolution. But ſuch is the fallacy of this hypotheſis, that it renders Mr. SIMPSON's concluſions obviouſly wrong, though his reaſoning is perfectly correct: for it cannot ſurely be an *equal* chance *in all caſes* that one life ſhall die after another. In the ſhort term of a ſingle year the chances are indeed ſo nearly equal, that it would be wrong to perplex the ſolution by attempting greater accuracy. But when the number of years, and the difference between the ages of the two lives are, conſiderable, thoſe chances muſt vary in proportion; and, therefore, unleſs the contingency is blended with another which ſhall very much diminiſh the probability of the event, the ſolution, by thus indiſcriminately ſuppoſing the chances to be equal, muſt be rendered extremely inaccurate. In Mr. SIMPSON's 36th problem the ſolution by this means appears to be abſurd: for, in the particular caſe in which C is the oldeſt of the three lives, the value of the reverſionary annuity becomes $= \frac{C - AC}{2}$; that is, the value of an annuity in this caſe during the life of C after B and A, provided A dies firſt, is the ſame whatever be the age of B; for no mention is made of his life in the foregoing expreſſion. It ſhould be obſerved, however, that the rule itſelf is ſtrictly true, and that the error ariſes from Mr. SIMPSON's having been miſled by the hypotheſis in determining the probability of B's dying after A in his inveſtigation of the 34th problem, which is applied to the ſolution of this problem[*].

I have declined giving ſpecimens of the different values of the reverſions as deduced from the foregoing rules and thoſe which have been hitherto publiſhed, not only from an apprehenſion of becoming tedious, but alſo from the conviction that at preſent they are unneceſſary; thoſe which I have formerly given being, I think, ſufficient to prove the inaccuracy of M. DE MOIVRE's hypotheſis. In thoſe inſtances in which I have compared ſome of the foregoing rules with the approximations now in uſe, I have invariably found the latter to be erroneous; nay, in ſome caſes, the values were almoſt twice as great as they ought to have been. This is particularly true when one of the lives is very young, and both or either of the other lives are very old. In reverſions of this kind I believe that this is generally the caſe, and that it ſeldom happens that the ages of the three lives are nearly equal. The approximations therefore can hardly ever be uſed with ſafety, and it will certainly be moſt prudent not to have recourſe to them when the correct values can be obtained. Should the difficulties attending the ſolution of the remaining problems which involve three lives be ſurmounted (and the taſk may not perhaps be impoſſible), the hypotheſis of an equal decrement of life, as far as it relates to any uſeful purpoſe in the doctrine of annuities, may then be totally abandoned. Or ſhould it even be found impracticable to deduce ſolutions of thoſe problems which are ſtrictly and accurately true; yet, I am ſatisfied from my own experience that ſuch near approximations may be procured as to render this hypotheſis equally unneceſſary.

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[*] It is proper to obſerve, that I have followed Mr. SIMPSON's method of determining this contingency in the 23d, 27th, 28th, and 29th Problems in my Treatiſe on Annuities.